Journal of Science and Technology 17 (June 2019): 111-118

THE TOWER OF BRAHMA WITH ONE EVILDOER: THE LIFE-TIME OF THE WORLD

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ABSTRACT

At the creation of the world, as the legend says, there was a Tower of Brahma with three diamond poles and 64 gold discs, in a temple. The priests there are in the process of transferring the tower from one pole to another, in minimum number of moves, where each move transfers one disc from one pole to another under the divine rule that no disc can ever be placed directly on top of a smaller one. As soon as the task of the priests would be completed, the world would come to an end. Single relaxation of divine rule was addressed previously. This paper considers the case when there is an evildoer disc which can be placed on top of a smaller one any number of times.We find that an evildoer disc in the Tower of Brahma will take the world to an end earlier.

Keywords: Divine rule, evildoer, life-time of the world, tower of Hanoi

INTRODUCTION

In 1885, the famous French number theorist, Francois Edouard Anatole Lucas(1842–1891), introduced a new mathematical puzzle in Europe, which became very popular during that time (Lucas (1885)). The toy puzzle is as follows: Given are three pegs, *S*, *P* and *D*, and *8* discs of different sizes. Initially, the discs rest on the peg, *S*, in a *tower* in small-on-large ordering (with the largest disc at the bottom, the second largest above it, and so on, and the smallest disc at the top), as shown in Figure 1. The problem is to transfer this tower from the peg *S* to the peg *D*, in original ordering, in minimum number of moves, where each move can transfer the topmost disc from one peg to another under the "divine rule", which demands that, at any stage of the transfer process,no disc can ever be placed on top of a smaller one.To market the puzzle, the following legend was attached to it (Ball(1892) and Gardner (1956)):

"In the great temple of Benares beneath the dome, which marks the center of the world, rests a brass plate in which are fixed three diamond needles, each a cubit (*45–56* cm) high and as thick as the body of a bee. On one of these needles, at the creation, God placed *64* discs of pure gold, in a tower, the largest disc resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the *Tower of Brahma*. Day and night unceasingly, the priests transfer the discs from one diamond needle to another according to the fixed and immutable laws of Brahma, which require that the priest on duty must not move more than one disc at a time and that he must place this disc on a needle so that there is no smaller disc below it. When all the *64* discs shall have been transferred from the needle on which God

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placed them to one of the other needles, tower, temple and Brahmins alike will crumble into dust, and with a thunderclap the world would vanish".

In the more general setting, the problem, called the *Tower of Hanoi problem*,is as follows: Given are three pegs, *S*, *P* and *D*, and *n* ($\geq$1) discs of different radii, $D_1, D_2, ..., D_n$, in increasing order. Initially, the discs are rest on the *source peg*, *S*, in a *tower* in small-on-large ordering as shown in Figure 2. The objective is to transfer this tower from the peg, *S*, to the *destination peg*, *D*, in minimum number of moves (using the available pegs), where each move can transfer one (the topmost) disc from one peg to another, under the "divine rule" that no disc can ever be placed on a smaller one.Let $M(n)$ denote the minimum number of legal moves required to solve the above problem. Then, the expression of $M(n)$is given as follows-

Theorem 1:$M(n)=2^n - 1,\ n\geq 1$.

Chen *et al*. (2007) proposed a variant with an evildoer disc which can be placed on top of a smaller disc any number of times. In this paper, we have studied the effect of the evildoer on the life-time of the world. Also we have enriched the calculation process to find out the minimum number of moves of the Tower of Hanoi problem with an evildoer. These are presented in the results and discussion part of this paper. Materials and methods section consists of thebackground materials and the research process of this study. Conclusions part summarizes the research output and limitation, and provides the future research direction.

MATERIALS AND METHODS

A new variant of the Tower of Hanoi problem was offered by Chen*et al*. (2007), which allows one or more relaxation of the "divine rule". In the more general setting, the modified problem is as follows. Given are the three pegs, *S*, *P* and *D*, and *n* ($\geq$1) discs of different sizes, resting on the source peg *S* in a tower, the problem is to shift the tower from the peg *S* to the peg *D*, in minimum number of moves, where each move can transfer only the topmost disc from one peg to another such that no disc is ever placed on top of a smaller one, except only once. Let $S(n)$ denotes the minimum number of moves required to solve the modified problem, assuming that the second largest disc violates the divine rule once. Then, we have the following result, due to Chen*et al*. (2007), giving an expression for $S(n)$.

Theorem 2: For any $n\geq 1, S(n)=\begin{cases} 2n-1, & if\ 1\leq n\leq 3 \\ 2^{n-2}+5, & if\ n\geq 4 \end{cases}$.

Recall that, for the problem with *n* ($\geq$4) discs and one relaxation of the "divine rule", the optimal policy is as follows. First, transfer the tower of the topmost *n*–3discs, $D_1, D_2, ..., D_{n-3}$, (from the peg *S*) to the peg *P* (in $2^{n-3}-1$ moves), next shift the disc D_{n-2}(from the peg *S*) to the peg *D*, then move the disc D_{n-1} (from the peg *S*) to the peg *P* on top of the tower of *n*–3 discs (violating the "divine rule"), followed by the transfer of the disc D_{n-2} (from the peg *D*) to the peg *P*. After transferring the largest disc to the peg *D*, first shift the disc D_{n-2} (from the peg *P*) to the peg *S*, then move the disc D_{n-1} (from the peg *P*) to the peg *D*, next transfer the disc D_{n-2} (from the peg *S*) to the peg *D* and finally, move the tower of *n*–3discs (from the peg *P*) to the destination peg *D* to complete the tower on it. Thus, the total number of moves involved is

$$2[(2^{n-3}-1)+1+1+1]+1=2^{n-2}+5.$$

Detailed literature review and deep thinking pose better interpretation of the problem of Hanoi tower with an evildoer.

RESULTS AND DISCUSSION

Developments of theories are presented under the sub-heading “The Tower of Hanoi with an evildoer” and the interesting result is discussed under the sub-heading “The Life-time of the World”.

The Tower of Hanoi with an evildoer

Tower of Hanoi problem with an evildoer: Any of the n discsof the Tower of Hanoi is an evildoer, where the evildoer can be placed on top of a smaller disc any number of times.

Let $E(n)$ be the minimum number of moves required to solve the Tower of Hanoi problem with n discs and one evildoer. Based on computer search, Chen*et al*. (2007) found that

$E(n)=S(n)$, if $1\leq n\leq 7$,but$E(8)=57<S(8)$.

The above value of $E(8)$ is achieved by the scheme below,assuming that the third largest disc is an evildoer.

Step 1: Shift the tower of discs, D_1, D_2, D_3, $D_4$from the source peg S to the destination peg D, in $2^4-1=15$ moves,

Step 2: Move the disc $D_5$from the peg S to the peg P,

Step 3: Transfer the disc D_6 from the peg S to the peg D, violating the divine rule,

Step 4: Move the disc D_5 again from the peg P on top of the disc D_6,

Step 5: Shift the disc $D_7$from the peg S to the peg P,

Step 6: Move the disc $D_5$from the peg D to the peg P, on top of the disc D_7,

Step 7: Transfer the disc $D_6$from the peg D to the peg S,

Step 8: Move the tower of discs D_1, $D_2$from the peg D to the peg P in$2^2-1=3$ moves,

Step 9: Place the disc $D_6$from the peg S to the peg P,

Step 10: Move the tower of discs D_3, $D_4$from the peg D to the peg P, in 3 moves.

The total number of moves involved in the above 10 steps is 28. After 10 steps, the 7 discs are all on the peg P(not in a tower), and the destination peg D is empty.

Step 11: Move the largest disc $D_8$from the peg S to the peg D.

After *Step 11*, repeat the *Steps 1–10* in reverse order (with appropriate choices of the pegs) to get the tower on the peg D. More precisely, we follow the steps below.

Step 12: Shift the tower of discs D_3, $D_4$from the peg P to the peg S, in 3 moves,

Step 13: Transfer the disc $D_6$from the peg P to the peg D,

Step 14: Move the tower of discs D_1, $D_2$from the peg P to the peg S, in 3 moves,

Step 15: Shift the disc $D_6$from the peg D to the peg S, violating the divine rule,

Step 16: Transfer the disc $D_5$from the peg P to the peg S,

Step 17: Move the disc $D_7$from the peg P to the peg D,

Step 18: Move the disc $D_5$from the peg S to the peg P,

Step 19: Transfer the disc $D_6$from the peg S to the peg D, *Step 20:* Move the disc $D_5$from the peg P to the peg D,

Step 21: Finally, shift the tower of discs D_1, D_2, D_3, D_4 from the peg S to the peg D, in 15moves, to complete the tower on the destination peg.

From the above procedure, we observe that the number of moves is reduced in two ways. Firstly, by reducing the size of the tower of the topmost discs to be transferred in *Step 1* and secondly, by dividing the tower of four discs "efficiently", using the evildoer disc in *Steps 8–10*.

Lemma 1: For the modified problem with n discs, if the disc D_i (counted from the top of the tower of n discs) is an evildoer (for some integer i satisfying the condition $k+1 \le i \le n-2$), (in the sense that $E(n)<S(n)$), then the disc D_{i+1} (counted from the top in the tower of $n+1$ discs) is an evildoer for the modified problem with $n+1$ discs.

Proof: First, we consider the procedure of dismantling the topmost $n-1$ discs for the modified problem with n discs, and we confine our attention to the movements of the tower of the topmost k smallest discs, $D_1, D_2, ..., D_k$; $k<n-3$.

The transfer of this tower of k discs requires 2^k-1 number of moves. Subsequently, it is divided into two sub-towers, $T_1=\{D_1, D_2, ..., D_\ell\}$ and$T_2=\{D_{\ell+1}, D_{\ell+2}, ..., D_k\}$, which are then moved, using the evildoer, in $2^\ell-1$ and $2^{k-\ell}-1$ number of moves respectively. Let the next $n-k-1$ largest discs require N number of moves. Thus, the total number of moves required to dismantle the initial tower of the topmost $n-1$ discs on the source peg S (just before the transfer of the largest disc D_n) is

$$2^k+2^\ell+2^{k-\ell}+N-3,$$

and by assumption,

$$2(2^k+2^\ell+2^{k-\ell}+N-3)+1<2^{n-2}+5=S(n)$$

that is,

$$2^{k+1}+2^{\ell+1}+2^{k-\ell+1}+2N<2^{n-2}+10. \quad (1)$$

Now, we consider the problem with $n+1$ discs. Let the disc D_{i+1} be an evildoer.To dismantle the topmost n(of the $n+1$) discs, we first transfer the topmost $k+1$ discs in $2^{k+1}-1$ number of moves. Subsequently, this tower of $k+1$ discs is divided into two sub-towers. Without loss of generality, the sub-towers may be chosen as $\{D_1, D_2, ..., D_{\ell+1}\}$ and $\{D_{\ell+2}, D_{\ell+2}, ..., D_{k+1}\}$. Using the evildoer disc D_{i+1}, these two sub-towers are moved in $2^{\ell+1}-1$ and $2^{k-\ell}-1$ number of moves respectively. Following the same procedure as that for the problem with n discs, the remaining $n-k-1$ discs require N number of moves. Thus, the total number of moves required is

$$2(2^{k+1}+2^{\ell+1}+2^{k-\ell}+N-3)+1$$

$$=2(2^{k+1}+2^{\ell+1}+2^{k-\ell}+N)-5. \quad (2)$$

Since $k<n-3$, it follows that $N \ge n-k-1>2$ (so that $N \ge 3$). Also, $2^{k-\ell} \ge 2$. Thus,

$$2^{k+1}+2^{\ell+1}+2^{k-\ell+1}+2N \ge 2^{k+1}+2^{\ell+1}+2^{k-\ell}+N+5.$$

Combining this with (1), we get

$$2^{k+1}+2^{\ell+1}+2^{k-\ell+1}+N<2^{n-2}+5. \quad (3)$$

It then follows from (2) and (3) that

$2(2^{k+1}+2^{\ell+1}+2^{k-\ell}+N)-5<2^{n-1}+5=S(n+1)$,

showing that, treating the disc D_{i+1} as an evildoer, we can reduce the number of moves for the modified problem with n+1 discs.

Corollary 1: For all $n\geq8$, the disc D_{n-2} is the unique evildoer.

Proof: When n=8, the disc $D_6=D_{n-2}$ is the evildoer. The result now follows from Lemma 1.

An expression of $E(n)$ is given in Theorem 3. To prove it, we need the result below.

Lemma 2: For any integer $k\geq2$ fixed, consider the optimization problem:

$$F(k)=\min_{\substack{a+b=k\\ a,b\ \text{integers}}}\left\{2^a+2^b\right\}-1.$$

Then,

$$F(k)=\begin{cases}2^{m+1}-1, & \text{if } k=2m\\ 3.2^m-1, & \text{if } k=2m+1\end{cases}.$$

Proof: First, let k=2m for some integer $m\geq1$. Then, the objective function in Lemma 2 is minimized at (the unique point) $a=b=m$. Next, let $k=2m+1$. In this case, the objective function attains its minimum at $a=m$, $b=m+1$. We then get the desired result after simplification.

The expression of $E(n)$ is now given below.

Theorem 3: For $n\geq8$,

$$E(n)=\begin{cases}2^{2m-3}+2^m+9, & \text{if } k=2m\\ 2^{2(m-1)}+3.2^{m-1}+9, & \text{if } k=2m+1\end{cases}$$

with

$E(n)=S(n)$ *for* $1\leq n\leq7$.

Proof: To find the expression for $E(n)$, $n\geq8$, we follow the scheme below.

Step 1: Shift the tower of the topmost n–4 (smallest) discs $D_1, D_2, \ldots, D_{n-4}$ (from the peg S) to the peg D, in $2^{n-4}-1$ moves.

Step 2: Move the disc D_{n-3}(from the peg S) to the peg P.

Step 3: Shift the discD_{n-2} (from the peg S) to the peg D, violating the "divine" rule.

Step 4: Move the disc D_{n-3}(from the peg P) to the peg D.

Step 5: Transfer the disc D_{n-1} (from the peg S) to the peg P.

Step 6: Place the disc D_{n-3} (from the peg D) on the disc D_{n-1} on the peg P.

Step 7: Shift the disc D_{n-2} (from the peg D) to the peg S.

Step 8: Divide the tower of n–4 discs on the peg S into two subtowers, T_1 andT_2 (according to the criterion of Lemma 2), of sizes a and b respectively. Then, movethe subtower T_1 (from the peg D) to the peg P, next, place the evildoer D_{n-2} (from the peg S) on top of T_1, and finally, shift the subtower T_2 (from the peg D) to the peg P. This step requires $F(n-4)$ number of moves.

After *Step 8*, the $n–1$ discs are all on the peg P, the largest disc is on the peg S, and the peg D is empty.

Step 9: Move the largest disc, D_n, (from the peg S) to the peg D.

After transferring the largest disc to the peg D, the $n–1$ discs, now resting on the peg P, are moved to complete the tower on the peg D. To do so, we just repeat *Steps 1–8* in reverse order, with appropriate choices of the pegs.

The total number of moves required is

$$2[2^{n-4}+F(n-4)+5]+1=2^{n-3}+2F(n-4)+11.$$

Therefore,

$$E(n)= 2^{n-3}+2F(n-4)+11.$$

Now appealing to Lemma 2, we get the desired result.

When $n=8$ (so that $m=4$), Theorem 3 gives, $E(8)=57$, and when $n=9$ (so that$m=4$), $E(9)=97$.

The Life-time of the World

Using Theorem 1, we can estimate the life-time of the world as follows. Since

$$M(64)=2^{64}-1=18,446,744,073,709,551,615,$$

assuming that each move takes 1 second, the priests of the Temple would require more than 5.84×10^{11} years to complete their assigned task. If we know the current state of the discs on the three needles in the Temple of Benares, we can calculate the age of the world. It is claimed (see, Hinz*et al*. (2013), Exercise 2.8, pp. 129) that, poking at the Temple of Benares, it has been found that one of the three needles is empty, one needle contains the disc D_{59}only, and the disc D_{58} is in the hands of the priest. If this is correct, the priests are about to form the tower of size 59. To do so, they had to form of the tower of size 58 on the now empty needle, in $2^{58}-1$ number of moves, put the disc D_{59} in the third needle, and then transferred the tower of size 57 to the original source needle, in $2^{57}-1$ number of moves to free the disc D_{58}. Thus, the priests have already made

$$(2^{58}-1)+1+(2^{57}-1)=2^{58}+2^{57}-1$$

number of moves, that is, they have already passed 13.7 billion years in the process of transferring the tower of 64 discs from one needle to the goal needle. According to the modern theory of cosmology, though the age of the universe is about 13.8 billion years, our solar system is much younger, only 4.8 billion years. It seems that the intruder into the Temple at Benares made a mistake to assess the current state of the discs on the three needles!

According to the legend, the life-time of the world is about 5.84×10^{11} years. But what happens if God, in a Sermon, allows the priests one relaxation of the divine rule, so that, during the process of transferring the 64 discs from the source needle to the goal needle, only once the priests can break the divine rule by placing a larger disc directly on a smaller one. This problem has been considered earlier in Majumdar(2016).

Now, consider the case when an evildoer is allowed during the transfer process in the Temple. How would the life-time of the world change? Or, if God allows the priest to choose the evildoer, which disc should be chosen by the priests?

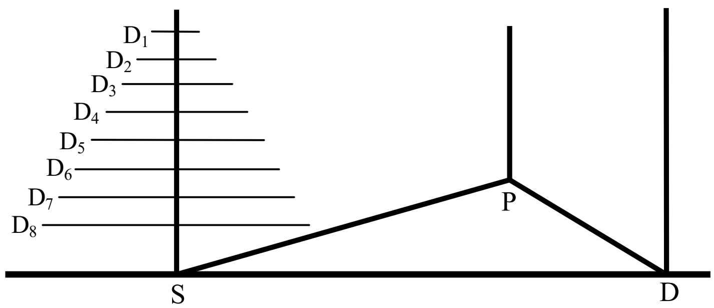

Figure 1. Initial stage of 8 discs Tower of Hanoi problem.

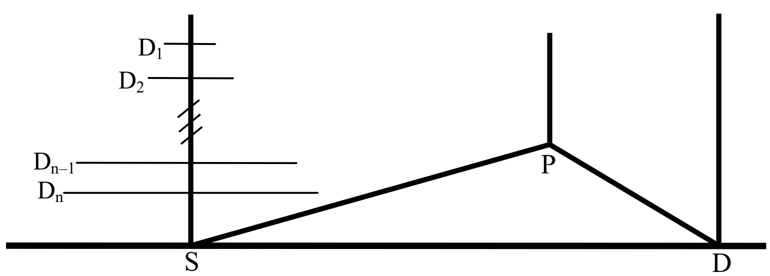

Figure 2. Primary phase of *n* discs Tower of Hanoi problem.

CONCLUSIONS

This study presents an efficient process to find out the minimum number of moves to solve the Tower of Hanoi problem with an evildoer disc. It also shows that the minimum number of moves ina single-evildoer Tower of Hanoi problem is attained if the third largest disc is an evildoer, i.e., (n–2)th disc is an evildoer in a problem of n discs. It is known that the relaxation of divine rule reduces the optimum number of moves. So, if the puzzle of the Tower of Brahma is tangible, then the evildoer disc will help the priests to finish their disc-transferring job quickly and the world will vanish with a thunderclap earlier than 5.84×10^{11} years. One can extend this study by considering 4 pegs in lieu of 3 pegs or by considering more than one evildoer.

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