Journal of Science and Technology 17 (June 2019): 119-125

ON THE SMARANDACHE RECIPROCAL PARTITION SETS

A.A.K. Majumdar[1] and S.M.S. Islam[2]*

[1]Ritsumeikan Asia-Pacific University, 1 – 1 Jumonjibaru, Beppu-shi 874-8577, Japan;
[2]Department of Mathematics, Hajee Mohammad Danesh Science and Technology University, Dinajpur-5200, Bangladesh

ABSTRACT

If the sum of the reciprocals of some distinct integer numbers is unity, then the set of those integers is called the Smarandache distinct reciprocal partition of unity. A prior research introduced this concept and denoted the Smarandache distinct reciprocal partition of unity in n partitions by SDRPS(n). This study has developed some methods to find out the Smarandache distinct reciprocal partition of unity, especially SDRPS(3) and SDRPS(4).

Keywords: Diophantine equation,Smarandache reciprocal partition of unity

INTRODUCTION

Murthy (2000) introduced the idea of the sets of Smarandache reciprocal partition of unity, which was studied in a systematic manner by Murthy and Ashbacher (2005). We start with the definition below, due to Murthy (2000).

Definition 1: The Smarandache distinct reciprocal partition of unity in n partitions, is denoted by SDRPS(n), and is defined by

$$\text{SDRPS}(n)=\{(a_1, a_2, \ldots, a_n): 0<a_1<a_2<\ldots<a_n;\ \frac{1}{a_1}+\frac{1}{a_2}+\ \ldots\ +\frac{1}{a_n}=1\}.$$

The order of the set SDRPS(n) is denoted by $f_{DR}(n)$.

Note that, the n (distinct) integers a_1, a_2, …,a_n can always be rearranged, if necessary, to satisfy the condition that $a_1<a_2<\ldots<a_n$. Thus,

$$\text{SDRPS}(2) = \{(a,b): \frac{1}{a}+\frac{1}{b}=1 \ and \ \ a<b\}.$$

Obviously, SDRPS(2)=∅ (the empty set).

The main objective of this study isto develop the method to calculate SDRPS(n).This paper also derives the expressions of SDRPS(3) and SDRPS(4). These are done in results and discussion part. Background materials and the method of this study are included in the materials and methods section.Conclusions portion concludes the research output and limitation, and provides the future research direction.

*Corresponding author: S.M. Shahidul Islam, Department of Mathematics, Hajee Mohammad Danesh Science and Technology University, Dinajpur-5200, Bangladesh, Cell Phone: 01718617334, E-mail: sislam.math@gmail.com

MATERIALS AND METHODS

Lemma 1: In the set SDRPS(n), $n \geq 3$, $a_1 \leq n - 1$, a_n is not a prime.

Proof: Let, on the contrary $a_1 \geq n$, so that

$$n \leq a_1 < a_2 < \ldots < a_n.$$

But then

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} < 1,$$

which contradicts the condition of the definition.

Next, let $a_n = p$, where p is a prime. Letting

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{n-1}} = \frac{A}{a_1 a_2 \ldots a_{n-1}}$$

We get

$$pA = (p-1)\, a_1 a_2 \ldots a_{n-1},$$

and we reach to a contradiction, since none of $a_1, a_2, \ldots, a_{n-1}$ is divisible by p.

Thus, instudyingSDRPS(n), it is sufficient to consider the case when $n \geq 3$. This background materials and deep thinking along with the literature review provide some methods to calculate SDRPS(n) given in the results and discussion section.

RESULTS AND DISCUSSION

This results and discussion section is divided into two sub sections namely “Main Results” and “Remarks”.

Main Results

First, we prove the following results.

Lemma 2: For $n \geq 3$, there always exist integers $a_1, a_2, \ldots, a_n$, satisfying the condition

$$2 = a_1 < a_2 < \ldots < a_n$$

such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1.$$

Proof: The proof is by induction on n.

When $n=3$, choosing

$$a_2 = 3,\ a_3 = 6,$$

we get

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = 1.$$

Thus, the result is true for $n=3$. To proceed by induction, we assume that the result is true for some integer n, that is, we assume that there are n integers $2=a_1, a_2, \ldots, a_n$ with

$$a_1 < a_2 < \ldots < a_n$$

such that

$$\frac{1}{a_1}+\frac{1}{a_2}+\ \dots\ +\frac{1}{a_n}=1\ .$$

To prove the result for n+1, we define the integers b_1, b_2, …,b_n, b_{n+1} as follows:

$b_1 = a_1$,

$b_2 = a_2$,

…

$b_{n-1} = a_{n-1}$,

$b_n = a_n + 1$,

$b_{n+1} = a_n(a_n + 1)$.

Clearly, $b_1 < b_2 < \dots < b_n < b_{n+1}$; moreover, b_1, b_2, …,b_n, b_{n+1} satisfy the condition:

$$\frac{1}{b_1}+\frac{1}{b_2}+\ \dots\ +\frac{1}{b_n}+\frac{1}{b_{n+1}}=\frac{1}{a_1}+\frac{1}{a_2}+\ \dots\ +\frac{1}{a_n}=1\ .$$

This proves the validity of the result for n+1, thereby establishing the lemma.

The result in Lemma 2 is interesting. It proves that, for any set SDRPS(n) for$n \geq 3$, we may have $a_1 = 2$. The proof of Lemma 2 gives a procedure of obtaining an element of the set SDRPS(n+1), starting with the element (a_1, a_2, …, a_n) of SDRPS(n).

To find SDRPS(3), we first note that, by Lemma 1, we must have $a_1 = 2$. We now prove the following result.

Lemma 3:a=3, b=6 is the only solution of the Diophantine equation

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{2}\,;\ 0<a<b. \qquad (1)$$

Proof: We recast the equation in (1) in the following equivalent form:

$2(a+b)=ab$.

Then, a must divide b, so that

$b=ka$ for some integer $k \geq 2$.

Therefore, we get

$2(1+k)=ka$.

Then, a must divide $1+k$ (since k does not divide $1+k$).

Now, when k=2, $a=k+1=3$ (and b=6), which we intended to prove.

Lemma 3, together with the fact that in SDRPS(3) a_1 cannot be greater than 2, proves the lemma below.

Lemma 4:SDRPS(3)={(2, 3, 6)} is the singleton set.

Next, we find SDRPS(4). To do so, we need some preliminary results given below.

Lemma 5: The only solutions of the Diophantine equation

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{6}\,;\ 0<a<b, \qquad (2)$$

are (i) a= 7, b=42, (ii) a=8,b=24, and (iii) a=9, b=18.

Proof: We rewrite (2) as

$$6(a+b)=ab.$$

Now, *a* must divide *b*, say

$$b=ka \text{ for some integer } k\geq 2.$$

Then

$$6(1+k)=ka.$$

Thus, *a* must divide 1+ *k* (since *k* does not divide 1+*k*), and *k* must divide 6.

Now, when *k*=2, *a*=3(*k*+1)=9 (and *b*=18), when *k*=3, then *a*=2(1+*k*)=8 (and *b*=24), and when *k*=6, a=1+*k*=7 (so that *b*=42). Hence, we get the desired result.

Lemma 6: The only solutions of the Diophantine equation

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{4}\text{ ; } 0<a<b, \qquad (3)$$

are (i) *a*= 5, *b*=20, and (ii) *a*=6,*b*=12.

Proof: Rewriting (3) as

$$6(a+b)=ab,$$

We see that, *a* must divide *b*, so that

$$b=ka \text{ for some integer } k\geq 2.$$

Then

$$4(1+k)=ka,$$

so that *a* must divide 1+ *k*, and 4 must be divisible by *k*.

Then, the desired solutions correspond to *k*=2 and *k*=4 respectively.

Lemma 7: The following Diophantine equation has no solution:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}\text{ ; } 5\leq a<b<c.$$

Proof: First, let *a*=5. Then,

$$\frac{1}{6}+\frac{1}{7}>\frac{3}{10}.$$

Since

$$\frac{1}{6}+\frac{1}{8}<\frac{3}{10},$$

it follows that *a*≠5. So, let *a*≥6. But then, for *c*>*b*>*a*≥6,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{1}{6}+\frac{1}{7}+\frac{1}{8}<\frac{3}{10}.$$

All these complete the proof of the lemma.

Lemma 8: The Diophantine equation below has no solution.

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{2}{3}\text{ ; } 4\leq a<b<c.$$

Proof: Since, for $c>b>a\geq 4$,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq \frac{1}{4}+\frac{1}{5}+\frac{1}{6}<\frac{2}{3},$$

the result follows.

The set SDRPS(4) is given in Lemma 9.

Lemma 9:SDRPS(4) is given as follows:

SDRPS(4)={(2, 3, 7, 42), (2, 3, 8, 24), (2, 3, 9, 18), (2, 3, 10, 15),
(2, 4, 5, 20), (2, 4, 6, 12)}.

Proof: To find SDRPS(4), first note that, by Lemma 1, a_1 is either 2 or 3. By Lemma 7, $a_1\neq 3$. Hence, a_1 must be 2. When $a_1=2$, by Lemma 7, a_2 cannot be greater than 5. Thus, the only possible values of a_2 are $a_2=3$, 4. With $a_1=2$, $a_2=3$, by Lemma 5, there are three solutions, and with $a_1=2$, $a_2=4$, by Lemma 6, there are two solutions.

It may be mentioned here that, in Murthy and Ashbacher (2005), SDRPS(3) is given without the proof that it is, in fact, a singleton set; also, only 5 elements of SDRPS(4) are listed there. We prove in Lemma 8 that SDRPS(4) has precisely 6 elements.

Remarks

Given the set SDRPS(n) (with $f_{DP}(n)$ elements), Murthy and Ashbacher (2005) considered the problem of extending it to get some of the elements of the set SDRPS(n+1). In this context, Murthy and Ashbacher (2005) suggest different methods. One such method is stated in Lemma 10, which is, in fact, due to Maohua (2001).

Lemma 10: Let $(a_1, a_2, \ldots, a_n)\in$SDRPS($n$). Then, $(2, 2a_1, 2a_2, \ldots, 2a_n)\in$SDRPS($n$+1).

Proof: Simply, we can prove it by showing the sum of the reciprocals of 2, $2a_1$, $2a_2$, …, $2a_n$ equal to 1.

We can also prove Lemma 11, which may be employed to find an element of the set SDRPS(n+2), starting with an element of SDRPS(n).

Lemma 11: Let $(a_1, a_2, \ldots, a_n)\in$SDRPS($n$). Then,

$(2, 3, 6a_1, 6a_2, \ldots, 6a_n)\in$SDRPS($n$+2).

Proof: By assumption,

$$2\leq a_1<a_2<\ldots<a_n,$$

with

$$\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}=1.$$

Therefore,

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6a_1}+\frac{1}{6a_2}+\ldots+\frac{1}{6a_n}=\frac{5}{6}+\frac{1}{6}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)=1.$$

Thus, the lemma is established.

Now, let $(a_1, a_2, \ldots, a_i, \ldots, a_n)\in$SDRPS($n$). Then, replacing a_i $(1\leq i\leq n)$ by b_{i1} and b_{i2}, where

$$b_{i1}=a_i+1,\ b_{i2}=a_i(a_i+1), \qquad (4)$$

we see that (a_1, a_2, …, a_{i-1}, b_{i1}, b_{i2}, a_{i+1}, …, a_n) (rearranging the numbers, if necessary),an element of SDRPS(n+1).

The third method suggested by Murthy and Ashbacher (2005) is as follows.Letd_{i1} and d_{i2} be two distinct divisors of a_i, so that $a_i=d_{i1}\times d_{i2}$. Then, replacing a_i by c_{i1} and c_{i2}, where

$$c_{i1}=d_{i1}(d_{i1}+d_{i2}),\ c_{i2}=d_{i2}(d_{i1}+d_{i2}),$$

(and rearranging the terms, if necessary), we get an element of SDRPS(n+1).

Thus, starting with (2, 3, 6)∈SDPRS(3), by applying the three methods outlined above, we get the following elements of SDRPS(4):

(2, 4, 6, 12), (2, 3, 7, 42), (2, 3, 10, 15).

This example shows that, the methods described above are overlapping, and do not generate all the elements of SDRPS(4).

To find a lower bound for $f_{DR}(n)$, we confine our attention to SDRPS(4) with 6 elements. A closer look at the elements of SDRPS(4) show that the second procedure described in (4) can be applied to only three (of the four components) in each of the first five elements of SDRSP(4). Then, by the second method, we get the following 17 elements of SDRPS(5): (2, 4, 7, 12, 42), (2, 3, 8, 42, 56), (2,3,7,43,1806), (2,4,8,12,24), (2,3,9,24,72), (2, 3, 8, 25, 600), (2,4,9,12,18), (2,3,10,18, 90), (2,3,9,19,342), (2,4,10,12,15), (2,3,11,15,110), (2,3,10,16,240), (3,4,5,6,20), (2,4,6,20, 30), (2,4,5,21,420), (2,4,6,12,20) and (2,4,6,13,156). Applying the third method, the following eleven elements of SDRPS(5) are obtained: (2,3,7,78,91), (2,3,8,33,88), (2,3,8,40,60), (2, 3, 8, 28, 168), (2,3,9,22,99), (2,3,9,27, 54), (2,3,14,15,35), (2,3,10,24,40), (2,4,5,24,120), (2, 4, 5, 36, 45), (2,4,6,21,28). And finally, by Lemma 10, the following four elements of SDRPS(5) result: (2,4,6,14,84), (2,4,6,16,48), (2,4,6,18,36) and (2, 4, 8, 10, 40). Thus, the suggested three methods together give only 32 elements of SDRPS(5). This leads to the following conservative estimate of SDRPS(n):

$$f_{DR}(n+1)\geq(n-1)[f_{DR}(n)-1]+n-2+f_{DR}(n)=nf_{DR}(n)-1.$$

In the above inequality, the number of elements of SDRPS(n+1), obtained from the elements of SDRPS(n) by the method of (4), is $(n-1)[f_{DR}(n)-1]+n-2$; and since a_n is not a prime, we may safely say that the number of elements of SDRPS(n+1), arising from the elements of SDRPS(n) by the third method, is $f_{DR}(n)$.

Considering, $A_1=\{(2, 3, 10, 15)\}$, $B_1=\{(4, 5, 6, 8, 9, 12, 20, 72)\}$,

we see that both A_1 and B_1 satisfy Definition 1 with $A_1\cap B_1=\varnothing$ and no common component. Then, applying the procedure outlined in (4) for the last components of A_1 and B_1, we get

$$A_2=\{(2, 3, 10, 16, 15\times16)\},\ B_2=\{(4, 5, 6, 8, 9, 12, 20, 73, 72\times73)\},$$

where$A_2\cap B_2=\varnothing$ with no components common, with $A_2\in$SDPRS(5) and $B_2\in$SDPRS(9). Applying the procedure (4) once more, we get

$$A_3=\{(2, 3, 10, 16, 240, 240\times241)\},\ B_3=\{(4, 5, 6, 8, 9, 12, 20, 73, 5257, 5256\times5257)\},$$

with$A_3\cap B_3=\varnothing$ with no common components and each of A_3 and B_3 satisfying (4). Continuing the process, we get two infinite sequences of sets $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ such that each A_i and B_i satisfies the condition in Definition 1with$A_i\cap B_i=\varnothing$ and no common components for any $i\geq1$. This example proves the conjecture of Murthy and Ashbacher (2005) that there are infinitely many disjoint sets A_i and B_i satisfying the condition of Definition 1.

CONCLUSIONS

This study has presented some methods to calculate Smarandache distinct reciprocal partition of unity in n partitions, SDRPS(n). The method by induction is the main process to find out SDRPS(n). We have found that SDRPS(3)={(2, 3, 6)} is the singleton set andSDRPS(4)consists of 6 elements such as (2, 3, 7, 42), (2, 3, 8, 24), (2, 3, 9, 18), (2, 3, 10, 15), (2, 4, 5, 20) and (2, 4, 6, 12). Thirty-two elements of SDRPS(5) are resultant from different methods. Infinitely many disjoint sets are found thosesatisfy the condition of the Smarandache distinct reciprocal partition of unity. One can extend this study by developing a direct method to calculate SDRPS(n) in lieu of induction process.

REFERENCES

Maohua L. 2001. A Conjecture Concerning the Reciprocal Partition Theory.Smarandache Notions Journal. 12: 242–243.

Murthy A. 2000. Smarandache Reciprocal Partition of Unity Sets and Sequences, Smarandache Notions Journal. 11: 218–226.

Murthy A and Ashbacher C. 2005.Generalized Partitions and Some New Ideas on Number Theory and Smarandache Sequences.ProQuest, Hexis, Phoenix, USA.pp.1–219.