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ON THE SINNER'S TOWER OF HANOI

A.A.K. Majumdar[1] and S.M. Shahidul Islam[2]*

[1]APU, 1-1 Jumonjibaru, Beppu-shi 875-8577, Oita-ken, Japan; [2]Department of Mathematics, Hajee Mohammad Danesh Science and Technology University, Dinajpur 5200, Bangladesh

ABSTRACT

The sinner's tower is a variant of the Tower of Hanoi where s (≥ 2) violations of the "divine rule" are allowed. This paper revisits the problem, and finds alternative optimal schemes in some particular cases. Some observations are also noted. We also derive an alternative approach of solving the problem when $n \geq 2s+3$.

Key words: Optimal solution, sinner's tower, tower of Hanoi

INTRODUCTION

The Tower of Hanoi (ToH) problem, invented by Lucas (1883), in general form, is: There are n (≥ 1) discs, $d_1, d_2, \ldots, d_n$, of different sizes, and 3 pegs, S, P, and D. Initially, the discs rest on the source peg, S, in a tower in increasing order, with the largest disc at the bottom, the second largest disc above it, and so on, with the smallest disc at the top. The objective is to shift the tower from S to the destination peg, D, in minimum number of moves, where each move can transfer the topmost disc from one peg to another under the "divine rule", which requires that, at any stage of the transfer process, no disc can ever be placed on top of a smaller one.

Over the past years, the Tower of Hanoi (ToH) has seen many variations, for example, the three-in-a-row puzzle introduced by Scorer et al. (1944), the cyclic ToH due to Atkinson (1981), the ToH with parallel moves and cyclic parallel moves due to Wu and Chen (1992) and Wu and Chen (1993), and the bottleneck ToH proposed by Wood (1981) and later treated by Poole (1992). Some of the variants of the ToH have been reviewed by Majumdar (2013) and Islam (2014). Recently, Chen et al. (2007) have introduced a new variant which allows s (≥ 1) violations of the "divine rule". The relaxation may be viewed as the sinner's s number of cheats (Chen et al. 2007), or alternatively, as the s especial rewards for the enlightened monk. Thus, in the new variant, the problem is to shift the tower from the peg S to the peg D in minimum number of moves, where for (at most) s moves, some disc may be placed directly on top of a smaller one.

Though the problem seems artificial because it actually allows much deeper relaxations (including "inverted tower") of the "divine rule", it is interesting to find that the problem has a closed-form solution.

Denoting by $S_3(n, s)$ the minimum number of moves required to solve the sinner's ToH with n discs s number of cheats, $S_3(n, s)$ is given as follows.

Proposition 1: For any $n \geq 1$ and s ($1 \leq s \leq n$), $S_3(n, s)$ is given by

$$S_3(n,s) = \begin{cases} 2n-1, & if \;\; n \leq s+2 \\ 4n-2s-5, & if \;\; s+2 \leq n \leq 2s+3 \\ 2^{n-2s}+6s-1, & if \;\; n \geq 2s+3 \end{cases}$$

*Corresponding author: Email: sislam.math@gmail.com, Cell phone: +8801718617334

For more detail, the readers are referred to the paper of Chen et al. (2007). It may be noted that the expression given above is slightly different from that of Chen et al. (2007); when $n = 2s + 3$, the problem can still be solved linearly (in n or s), as has been shown in the second section below.

This paper gives some alternative schemes in some particular cases, namely, when $n = s + 3$, $s + 4$, $s + 5$, $2s + 1$, $2s + 2$. However, for $n = 2s + 3$, the optimal scheme is unique. This is done in the materials and methods section. In the results and discussion section, we develop a formula satisfied by the optimal value function $S_3(n, s)$, which would help us understand the form of $S_3(n, s)$ when $n \geq 2s + 3$.

MATERIALS AND METHODS

In "proving" Proposition 1, Chen et al. (2007) give the schemes for all the three cases. However, in some particular cases, there are alternative schemes as well. These are given below.

Case 1: When $n = s + 3$.

In this case, another possible procedure is as follows.

1. Move the topmost two discs, d_1 and d_2, from the source peg, S to the (auxiliary) peg P in a tower, in 3 moves.
2. Transfer the next s discs, d_3, d_4, …, d_{s+2}, in this order, one by one (from the peg S) to the peg P in an "inverted tower", in s moves (violating the "divine rule" s times).
3. Shift the largest disc $d_n \equiv d_{s+3}$ from the peg S to the destination peg D.
4. Move the discs d_{s+2}, d_{s+1}, …, d_3 (in this order) one by one (from P) to D.
5. Transfer the discs d_1 and d_2 on the peg P to the peg D to complete the tower on D.

The total number of moves required is

$$2(3 + s) + 1 = 2s + 7.$$

Case 2: When $n = s + 4$.

The alternative scheme followed is as below.

1. Move the discs d_1 and d_2 from the peg S to the peg P in a tower.
2. Consider the pair of discs, (d_3, d_4). For this pair, d_3 is moved (from the peg S) to the peg D, followed by the transfer of the disc d_4 (from S) to P (which violates the "divine rule"), and then d_3 is shifted (from D) to P. This requires 3 moves.
3. With the $s-1$ discs d_5, d_6, …, d_{s+3} (on the peg S), form an "inverted tower" on the peg P (violating the "divine rule" $s-1$ times).
4. Transfer the disc $d_n \equiv d_{s+4}$ (from the peg S) to the peg D.
5. Move the discs d_{s+3}, d_{s+2}, … d_5 (on the peg P) one by one (in this order) to D.
6. Of the pair of discs (d_3, d_4) (on the peg P), d_3 is transferred to S, next d_4 is moved to D, and then d_3 is moved again (from S) to D.
7. Move d_1, d_2 (on the peg P) to the peg D.

The total number of moves involved in this scheme is

$$2[3 + 3 + (s - 1)] + 1 = 2s + 11.$$

Case 3: When $n = s + 5$.

Here, another possible scheme is as follows.

1. Move the discs d_1 and d_2 (from the peg S) to the peg P in a tower.
2. Form 2 pairs with discs, (d_i, d_{i+1}); $i = 3, 5$. For each pair (d_i, d_{i+1}), d_i is first moved (from S) to D, d_{i+1} is shifted (from S) to P (violating the "divine rule"), and then d_i is moved again (from D) to P. Each pair requires 3 moves. The "divine rule" is violated twice in this step.
3. Transfer (from the peg S) the $s - 2$ discs d_7, d_8, …, d_{s+4} one by one to form an "inverted tower" on the peg P (thereby violating the "divine rule" $s-2$ times).

4. Shift the disc $d_n \equiv d_{s+5}$ (from the peg S) to the peg D.
5. Move the discs d_{s+4}, d_{s+3}, … d_7 one by one (in this order) from P to D.
6. Of the pair of discs (d_i, d_{i+1}); i = 3, 5 (on the peg P), d_i is moved to the peg S, next d_{i+1} is moved to the peg D, and subsequently d_3 is moved (from S) to D.
7. Move d_1, d_2 (from the peg P) to the peg D.

The scheme requires

$$2[3+2\times 3+(s-2)]+1=2s+15$$

number of moves.

Case 4: When $n = 2s + 1$.

In this case, another optimal scheme is as follows.

1. Move the discs d_1 and d_2 from the peg S to the peg P in a tower.
2. Of the $2s-1$ discs remaining on S, form $s-2$ pairs with $2(s-2)$ discs (d_i, d_{i+1}); i = 3, 5, …, $2s-3$. For each pair (d_i, d_{i+1}), d_i is shifted to the peg D, next d_{i+1} is moved to P (violating the "divine rule"), and then d_i is transferred (from D) to P. Here, the total number of moves is $3(s-2)$, and the total number of violations of the "divine rule" is $s-2$.
3. Move (from the peg S) the discs d_{2s-1} and d_{2s} one by one (in this order) to form an "inverted tower" on the peg P (thereby violating the "divine rule" 2 times).
4. Shift the disc $d_n \equiv d_{2s+1}$ (from the peg S) to the peg D.
5. Transfer the discs d_{2s} and d_{2s-1} (in this order) from the peg P to D.
6. For each pair (d_i, d_{i+1}); $i = 2s-3$, …, 5, 3 (on P), d_i is moved to S, then d_{i+1} is shifted (from P) to D, followed by the movement of d_i is moved (from S) to D.
7. Move the discs d_1, d_2 (from the peg P) to the peg D.

The total number of moves needed is

$$2[3+3(s-2)+2]+1=6s-1.$$

Case 5: When $n = 2s + 2$.

In this case, an alternative scheme is given below.

1. Move the discs d_1 and d_2 (from the peg S) to the peg P in a tower.
2. Of the $2s$ discs left on S, form $s-1$ pairs with $2(s-1)$ discs, of the forms (d_i, d_{i+1}); i = 3, 5, …, $2s-1$. For each pair (d_i, d_{i+1}), d_i is moved to the peg D, next d_{i+1} is shifted to P (violating the "divine rule"), and then d_i is moved again (from D) to P. Here, the total number of moves is $3(s-1)$ with $s-1$ violations of the "divine rule".
3. Move the disc $d_{n-1} \equiv d_{2s+1}$ (from S) to P (violating the "divine rule").
4. Shift the disc $d_n \equiv d_{2s+2}$ (on S) to D.
5. Transfer the disc d_{n-1} from P to D.
6. For each pair of discs (d_i, d_{i+1}); $i = 2s-1$, …, 5, 3 (on P), d_i is shifted to S, followed by the transfer of d_{i+1} to D, and then d_i is moved again (from S) to D.
7. Move the discs d_1, d_2 (on the peg P) to the peg D.

The total number of moves required under this scheme is

$$2[3+3(s-1)+1]+1=6s+3.$$

Case 6: When $n = 2s + 3$.

Here, the transfer of the tower from the peg S to the peg D may be affected as follows.

1. Move the discs d_1 and d_2 (from S) to the peg P in a tower.
2. $2s + 1$ discs are left on S. Form s pairs (d_i, d_{i+1}); i = 3, 5, …, $2s+1$. Of the pair (d_i, d_{i+1}), d_i is first shifted to D, then d_{i+1} is transferred to the peg P (violating the "divine rule"), and then d_i is moved again (from D) to P. This step requires $3s$ moves with s violations of the "divine rule".

3. Shift the disc $d_n \equiv d_{2s+3}$ (from S) to D.
4. For each pair of discs (d_i, d_{i+1}); $i = 2s+1, \ldots, 5, 3$ (on P), d_i is moved to S, next d_{i+1} shifted to D, and then d_i is moved again (from S) to D.
5. Finally, move the discs d_1, d_2 (from the peg P) to the peg D.

This scheme requires, in total,

$$2(3+3s)+1=6s+7$$

number of moves.

In Case 6, let the scheme below be followed:

1. Move the three discs d_1, d_2 and d_3 from the peg S to the peg P in a tower, in 7 moves.
2. With the $2s$ discs remaining on S, form $s-1$ pairs (d_i, d_{i+1}); $i = 4, 6, \ldots, 2s$. For each pair (d_i, d_{i+1}), d_i is first moved to the peg D, next d_{i+1} is shifted to P (violating the "divine rule"), and then d_i is moved again (from D) to P. This step requires $3(s-1)$ moves with $s-1$ violations of the "divine rule".
3. Shift the disc $d_{n-1} \equiv d_{2s+2}$ (from S) to P (violating the "divine rule").
4. Shift the disc $d_n \equiv d_{2s+3}$ (from S) to D.
5. Shift the disc $d_{n-1} \equiv d_{2s+2}$ (from P) to D.
6. Of the pair (d_i, d_{i+1}); $i = 2s, \ldots, 6, 4$ (on P), d_i is moved to S, followed by the movement of d_{i+1} to D, and then d_i is moved again (from S) to D.
7. Move the discs d_1, d_2 and d_3 (from P) to D.

Under the above scheme, the total number of moves required is

$$2[7+3(s-1)+1]+1=6s+11,$$

so that this scheme is worse than the first one.

This shows that, for $n = 2s + 3$, there is only one optimal scheme.

From the analyses of Cases 1 – 6, we observe that, for $s+2 \le n \le 2s+3$, there is a common pattern of the configuration of discs on the peg P just before the movement of the largest disc. This is as follows:

Sub-configuration 1: discs d_1 and d_2 at the bottom.

Sub-configuration 2: pair(s) of discs (d_i, d_{i+1}) with d_i above d_{i+1}.

Each such pair requires *3* moves, and violates the "divine rule" once.

Sub-configuration 3: "inverted tower" of size k (violating the "divine rule" k times).

It may be noted here that the "inverted tower" is more economical (in terms of the number of moves) than the Sub-configuration 2, but given the number of violations of the "divine rule", the latter can handle more discs. There are two extreme cases: In Case 1, we have only Sub-configuration 1 and Sub-configuration 3; and in Case 6, we have only Sub-configuration 1 and Sub-configuration 2.

The analyses of Cases 1 – 5 shows that, in each case, in addition to the scheme given by Chen et al. (2007), there is an alternative scheme, each requiring the same number of moves. However, in proving Proposition 1 for $s+2 \le n \le 2s+3$, the expression given therein is helpful.

From the analysis of Case 6, we see that when $n = 2s + 3$, the problem can still be solved linearly (in n or s). Furthermore, the scheme is unique.

In this connection, we observe that, when $n = 2s+3$, we have the "saturated case" in the sense that all the s number of violations of the "divine rule" is exhausted in Step 2 of the scheme given in Case 6. This suggests that, if n increases further, we have to increase accordingly the number of discs to be moved on the peg P in Step 1 (given in Case 6).

RESULTS AND DISCUSSION

An alternative approach of solving the sinner's Tower of Hanoi problem was developed by finding an optimality equation satisfied by $S_3(n, s)$ when $n \geq 2s + 3$. As has already been mentioned, when $n \leq 2s + 3$, the problem can be solved linearly (both in n and s). It is thus sufficient to consider the case when $n \geq 2s + 3$.

To find the optimality equation satisfied by $S_3(n, s)$, $n \geq s + 1$, we proceed as follows:

1. Move the topmost k discs from the peg S to the peg P, in a tower, in $2^k - 1$ moves,
2. With the next $2l$ discs on S, form l pairs (d_i, d_{i+1}). For each pair (d_i, d_{i+1}), d_i is first moved to the peg D, next d_{i+1} is shifted to P (violating the "divine rule"), and then d_i is moved again (from D) to P. This step requires $3l$ moves, and the "divine rule" is violated l times.
3. Move the next m discs from S to P, in an "inverted tower", in m moves, violating the "divine rule" m times.
4. Transfer the largest disc d_n (from S) to D.
5. The m discs in the "inverted tower" on P are shifted, one by one, to D.
6. For each of the next l pairs of discs (d_i, d_{i+1}) on P, d_i is moved to S, next d_{i+1} is shifted to D, and then d_i is moved again (from S) to D.
7. Move the k discs from P to D, in a tower, to complete the tower on the peg D.

The total number of moves involved in the above 7 steps is

$$2\{(2^k - 1) + 3l + m\} + 1 = 2^{k+1} + 6l + 2m - 1,$$

and the total number of violations of the "divine rule" is $l + m$, where k (≥ 0), l (≥ 0), and m (≥ 0) are to be found out so as to minimize the total number of moves.

Thus, we get

$$S_3(n, s) = \min\{2^{k+1} + 6l + 2m - 1\} \tag{1}$$

such that

$$k + 2l + m = n - 1$$
$$l + m = s$$
$$k \geq 0,\ l \geq 0,\ m \geq 0.$$

From the two equality constraints, we get after eliminating l,

$$k = n - 2s - 1 + m. \tag{2}$$

We re-write the objective function in (1) as follows, using the constraint conditions $l + m = s$ and (2):

$$\begin{aligned} 2^{k+1} + 6l + 2m - 1 &= 2^{k+1} + 6(s - m) + 2m - 1 \\ &= 2^{k+1} + 6s - 4(k - n + 2s + 1) - 1 \\ &= 2^{k+1} - 4k + 4n - 2s - 5. \end{aligned} \tag{3}$$

Now, if $n = 2s + 1$, by virtue of (3), (1) reads as

$$S_3(2s + 1, s) = \min\{2^{k+1} + 6s - 4k - 1\},\ k \geq 0, \tag{4}$$

and it is easy to check that the minimum is attained at the two values $k = 1, 2$. Again, if $n = 2s + 2$, then (1) takes the form

$$S_3(2s + 2, s) = \min\{2^{k+1} + 6s - 4k + 3\},\ k \geq 1, \tag{5}$$

and the minimum is attained at $k = 1, 2$. From (4) and (5), we see that, for each of the cases $n = 2s + 1$ and $n = 2s + 2$, we have alternative schemes, as has been observed in the second section. On the other hand, if $n \geq 2s+3$, then $k \geq 2$, and the objective function (3) is strictly increasing in k, and hence, it is minimized at a unique point. In such a case, we must have $m = 0$ (so that $l = s$), and consequently,

$$S_3(n, s) = 2^{n-2s} + 6s - 1, \; n \geq 2s + 3.$$

It is an interesting problem to study the equation (1) in more detail. Chen et al. (2007) used induction on n to prove Proposition 1. However, to prove it for $n \geq 2s + 3$ by induction on n, the basis for induction should be $n = 2s + 3$, that is, it is necessary to show that

$$S_3(n, s) = 2^{n-2s} + 6s - 1, \text{ if } n = 2s + 3,$$

because the functional form is different for $n \geq 2s + 4$. But the more important fact is that, in proving Proposition 1 for $n \geq 2s + 4$ by induction on n, Chen et al. (2007) assume that the largest $n-1$ discs requires $2^{n-2s-1} + 6s - 1$ moves (and the smallest disc requires 2^{n-2s-1} moves), which has not been proved. Without settling these points, the proof of Proposition 1 (by induction) would remain incomplete.

On the other hand, the alternative approach, given in the third section, explains the case when $n \geq 2s + 4$.

Of particular interest is the case when $s = 1$, which is given below.

Corollary 1: For any $n \geq 1$,

$$S_3(n) = \begin{cases} 2n-1, & \text{if } 1 \leq n \leq 3 \\ 2^{n-2} + 5, & \text{if } n \geq 4 \end{cases}$$

We recall that, to market the Tower of Hanoi puzzle, the following legend was attached to it: At the creation of the world, there was a Tower of Hanoi with three (diamond) poles and *64* (gold) discs in a temple of Benares. The priests there are in the process of transferring the tower from one pole to another, in minimum number of moves, where each move can transfer one disc from one pole to another under the “divine rule” that no disc can ever be placed on top of a smaller one. As soon as the priests complete their task, the world would come to an end. According to this legend, the life-time of the world is some 5.84×10^{11} years, assuming that each move takes 1 second (see, for example, Ball (1892) and Gardner (1956)). What happens if a single relaxation of the “divine” rule is allowed? The problem has been treated by Majumdar (2016) on the basis of Corollary 1.

We conclude the paper with two open problems. The first problem is due to Chen et al. (2007).

ToH with k Evildoers: Any s (of the n) discs are evildoers, where each evildoer can be placed on top of a smaller disc any number of times.

Let $E_3(n, s)$ be the minimum number of moves required to solve the problem above. The objective is to find a formula for $E_3(n, s)$. Chen et al. (2007) report that,

$$E_3(n, 1) = S_3(n, 1) \text{ for } 1 \leq n \leq 7,$$

But $E_3(8, 1) = 57$ when the disc d_6 is chosen as the evildoer. Clearly, the expression for $E_3(n, s)$ depends on the choice of the evildoer disc.

Reve’s Puzzle with s Relaxations: In the Reve’s puzzle (Dudeney (1958)), there are four pegs, S, P_1, P_2 and D. Initially, the n discs rest on S in a tower. The problem is to shift the tower (from S) to D in minimum number of moves, when s (≥ 1) violations of the “divine rule” are allowed. Denoting by $S_4(n, s)$ the minimum number of moves, it is interesting to find a formula for $S_4(n, s)$.

In a recent paper, Majumdar and Islam (2018) considered the problem of finding $E_3(n, 1)$. The explicit form is given below.

Lemma 1: For $n \geq 8$,

$$E_3(n, 1) = \begin{cases} 2^{2m-3} + 2^m + 9, & \text{if } k = 2m \\ 2^{2(m-1)} + 3.2^{m-1} + 9, & \text{if } k = 2m+1 \end{cases}$$

with

$E_3(n, 1) = S_3(n)$ for $1 \le n \le 7$.

In another paper, Majumdar (2019) has solved the Reve's puzzle with single relaxation of the "divine rule". Denoting by $S_4(n)$ the minimum number of moves required to solve the problem with n (≥ 1) number of discs, $S_4(n)$ is given as follows.

Lemma 2: For $n \ge 1$,

$$S_4(n) = \begin{cases} 2n-1, & \text{if } 1 \le n \le 4 \\ M_4(n-1)+2, & \text{if } 4 \le n \le 7 \\ M_4(n-2)+6, & \text{if } n \ge 7 \end{cases}$$

where $M_4(n)$ is the minimum number of moves required to solve the Reve's puzzle with n (≥1) number of discs.

It thus remains to solve the Reve's puzzle with s (>1) number of relaxations of the "divine rule".

CONCLUSION

This study has considered the sinner's tower of Hanoi where s (≥ 2) violations of the divine rule are allowed. Depending upon n, the number of discs, we have provided some alternative schemes for sinner's tower of Hanoi problem in some particular cases, namely, when $n = s+3$, $s+4$, $s+5$, $2s+1$, $2s+2$. Unique optimal scheme is found for the case of $n = 2s + 3$. For the future research direction, two open problems related to sinner's tower of Hanoi are discussed.

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