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A STUDY ON THE PROPERTIES OF SMARANDACHE SUMS OF PRODUCTS

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ABSTRACT

This study concentrates to discuss and illustrate (i) the properties of S(n, r), the general Smarandache sums of products of numbers, and (ii) Smarandache sums of products of odd and even natural numbers. Finally, it includes new properties, recurrence formulae related to Smarandache sums of products of numbers with numerical demonstration.

Key words: Generating function, natural number, recurrence relation, smarandache sum of products

INTRODUCTION

Smarandache number system is a new branch of number theory. The development of different properties of Smarandache sums of products can be applied to explicate some unexplained problems. This study deals with the Smarandache sums of products of natural numbers. Many interesting results related to this summation are obtained but some of them are still unproven. We have proved some of them in this paper. The Smarandache sum of products is defined by Vyawahare (2006) as follows.

Definition 1: For any integer $n \geq 1$, the Smarandache sum of products, denoted by $S(n, r)$, is the sum of the products of the first n natural numbers, taken r at a time, without repetition, where r is an integer with $0 \leq r \leq n$, and $S(n, 0) = 1$.

Example 1: For $n = 5$, we get the following functions for different values of r.

$S(5, 1) = 1 + 2 + 3 + 4 + 5 = 15$,

$S(5, 2) = 1.2 + 1.3 + 1.4 + 1.5 + 2.3 + 2.4 + 2.5 + 3.4 + 3.5 + 4.5 = 85$,

$S(5, 3) = 1.2.3 + 1.2.4 + 1.2.5 + 1.3.4 + 1.3.5 + 1.4.5 + 2.3.4 + 2.3.5 + 2.4.5 + 3.4.5 = 225$,

$S(5, 4) = 1.2.3.4 + 1.2.3.5 + 1.2.4.5 + 1.3.4.5 + 2.3.4.5 = 274$,

$S(5, 5) = 1.2.3.4.5 = 5!$.

Some elementary properties of $S(n, r)$, due to Ramsubramanian (1991), Islam (2014) and Majumdar and Islam (2018), are given below.

Lemma 1: For any integer $n \geq 1$,

(i) $S(n, n) = n!$, so that $S(n, n)$ satisfies the recurrence relation $S(n+1, n+1) = (n+1)\, S(n, n)$ for any integer $n \geq 0$,

(ii) $S(n, 1) = \dfrac{n(n+1)}{2}$ for any integer $n \geq 1$,

(iii) The number of terms in $S(n, r)$ is ${}^{n}C_{r}$.

Theorem 1: For any integer $n \geq 1$, the generating function of $S(n, r)$, denoted by $G_n(x)$, is given by

$$G_n(x) \equiv (x+1)(x+2)\ldots(x+n) = \sum_{r=0}^{n} S(n,r)x^{n-r} = \sum_{r=0}^{n} S(n,n-r)x^{r} . \quad (1)$$

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The generating function $G_n(x)$ satisfies the following relationships.

Corollary 1: For any integer $n \geq 1$, $\sum_{r=0}^{n} S(n,r) = (n+1)! = S(n+1, n+1)$.

Proof: Substituting $x = 1$ in equation (1), we get the desired result.

Corollary 2: $G_n'(x) = G_n(x)\left(\frac{1}{x+1} + \frac{1}{x+2} + \ldots + \frac{1}{x+n}\right)$ for any integer $n \geq 1$.

Proof: Logarithmic differentiation of $G_n(x) \equiv (x+1)(x+2) \ldots (x+n)$ gives the result.

Corollary 3: For any integer $n \geq 1$, the quantities $S(n, r)$ are given by

$$S(n, n-l) = \frac{1}{l!}\left[\frac{d^l}{dx^l} G_n(x)\right]_{x=0} ;\ 0 \leq l \leq n.$$

Proof: lth derivative of equation (1) with respect to x gives

$$\frac{d^l}{dx^l} G_n(x) = \sum_{r=l}^{n} S(n, n-r) \frac{r!}{(r-l)!} x^{r-l};\ 0 \leq l \leq n$$

Now putting $x = 0$, we get the result desired.

The main objective of this study is to prove some unproven theorems related to Smarandache sums of products of natural numbers. This paper also derives some new properties of Smarandache sums of products.

MATERIALS AND METHODS

Vyawahare (2006) also considered the sum of products of odd and even natural numbers, defined below.

Definition 2: For any integer $n \geq 1$,

(i) the Smarandache sum of products of odd natural numbers, denoted by $O(n, r)$, is the sum of the products of the first n odd natural numbers, taken r at a time without repetition,

(ii) the Smarandache sum of products of even natural numbers, denoted by $E(n, r)$, is the sum of the products of the first n even natural numbers, taken r at a time without repetition,

where r is an integer with $0 \leq r \leq n$, and $O(n, 0) = 1$, $E(n, 0) = 1$.

Clearly,

$$O(n, n) = 1.3 \ldots (2n-1) = \frac{(2n)!}{2^n n!};\ n \geq 1, \tag{2}$$

$$E(n, n) = 2.4 \ldots (2n) = 2^n n!;\ n \geq 1. \tag{3}$$

From equation (2) and (3), we see that $O(n+1, n+1)$ and $E(n+1, n+1)$ satisfy the following recurrence relations.

$$O(n+1, n+1) = (2n+1)\, O(n, n), n \geq 1, \tag{4}$$

$$E(n+1, n+1) = 2(n+1)\, E(n, n);\ n \geq 1. \tag{5}$$

The generating functions of $O(n, r)$ and $E(n, r)$ are given in the theorem below.

Theorem 2: For any integer $n \geq 1$,

(i) $$G_n^O(x) \equiv (x+1)(x+3) \ldots (x+2n-1) = \sum_{r=0}^{n} O(n,r) x^{n-r} = \sum_{r=0}^{n} O(n, n-r) x^r \tag{6}$$

(ii) $$G_n^E(x) \equiv (x+2)(x+4) \ldots (x+2n) = \sum_{r=0}^{n} E(n,r) x^{n-r} = \sum_{r=0}^{n} E(n, n-r) x^r \tag{7}$$

The following result is conjectured by Vyawahare (2006). We state it with a proof.

Lemma 2: For any integer $n \geq 1$,

(i) $E(n, n) = \sum_{r=0}^{n} O(n,r)$, **(ii)** $O(n+1, n+1) = \sum_{r=0}^{n} E(n,r)$.

Proof: The results follow from equations (6) and (7) respectively by substituting $x = 1$.

Vyawahare (2006) then proceeded to define the Smarandache generalized sum of products as follows.

Definition 3: For any integers $n \geq 1$ and $p \geq 0$, the Smarandache generalized sum of products, denoted by $S^{(p+1)}(n, r)$, is the sum of the products of n consecutive natural numbers, starting from the integer p, taken r at a time, without repetition, where r is an integer with $0 \leq r \leq n$, and $S^{(p+1)}(n, 0) = 1$.

Clearly, the generating function of $S^{(p+1)}(n, r)$ is as given in the following theorem.

Theorem 3: For any integers $n \geq 1$ and $p \geq 0$, the generating function of $S^{(p+1)}(n, r)$, denoted by $G_n^{(p+1)}(x)$, is given by

$$G_n^{(p+1)}(x) \equiv (x+p+1)(x+p+2)\ldots(x+p+n) = \sum_{r=0}^{n} S^{(p+1)}(n,r)x^{n-r} = \sum_{r=0}^{n} S^{(p+1)}(n,n-r)x^{r} \quad (8)$$

Obviously,

$$S^{(1)}(n, r) = S(n, r) \text{ for any integers } n \geq 1,\ 0 \leq r \leq n, \quad (9)$$

$$G_n^{(p+1)}(0) \equiv S^{(p+1)}(n, n) = \frac{(p+n)!}{p!} \text{ for any integers } n \geq 1, p \geq 0, \quad (10)$$

$$S^{(p+1)}(n+1, n+1) = (p+n+1)\, S^{(p+1)}(n, n) \text{ for any integers } n \geq 1, p \geq 0. \quad (11)$$

Equation (10) follows from equation (8) in the particular case when $x = 0$, and equation (11) follows from equation (10).

In this paper, we provide more recurrence relations satisfied by $S(n, r)$, $O(n, r)$, $E(n, r)$ and $S^{(p+1)}(n, r)$.

RESULTS AND DISCUSSION

$S(n, r)$ satisfies the following recurrence formula, which was mentioned, without any proof, by Vyawahare (2006).

Lemma 3: For any integers n and r with $n \geq r \geq 1$, $S(n+1, r) = S(n, r) + (n+1)\, S(n, r-1)$.

Proof: By Theorem 1,

$$(x+1)(x+2)\ldots(x+n)(x+n+1) = \sum_{r=0}^{n+1} S(n+1,r)x^{n+1-r}. \quad (12)$$

Now,

LHS of equation (12) = $(x+1)(x+2)\ldots(x+n)(x+n+1)$

$= (x+n+1) \sum_{r=0}^{n} S(n,r)x^{n-r}$ (using equation (1))

$= \sum_{r=0}^{n} S(n,r)x^{n+1-r} + (n+1)\sum_{i=0}^{n} S(n,i)x^{n-i}$

$= [S(n, 0)\, x^{n+1} + \sum_{r=1}^{n} S(n,r)x^{n+1-r}] + (n+1)[\sum_{i=0}^{n-1} S(n,i)x^{n-i} + S(n, n)]$

(putting $i = r - 1$ in the second sum)

$= [S(n, 0)\, x^{n+1} + \sum_{r=1}^{n} S(n,r) x^{n+1-r}\ ] + (n+1)[\sum_{r=1}^{n} S(n, r-1) x^{n+1-r} + S(n, n)]$

$= S(n, 0)\, x^{n+1} + \sum_{r=1}^{n} [S(n,r) + (n+1) S(n, r-1)] x^{n+1-r} + (n+1)\, S(n, n).$

Since $S(n + 1, 0) = S(n, 0) = 1$ and $(n + 1)\, S(n, n) = S(n + 1, n + 1)$ (by Lemma 1), we finally get,

$(x + 1)(x + 2) \ldots (x + n)(x + n + 1)$

$$= S(n+1, 0)\, x^{n+1} + \sum_{r=1}^{n} [S(n,r) + (n+1) S(n, r-1)] x^{n+1-r} + S(n+1, n+1). \qquad (13)$$

Now, comparing the coefficients of x^{n+1-r} from (12) and (13), we get the desired result.

Corollary 4: For all $n \geq r+1 > 1$,

$$S(n, r) = S(r, r) + \sum_{i=r}^{n-1} (i+1) S(i, r-1)$$

Proof: From Lemma 3, we have

$S(i+1, r) - S(i, r) = (i+1)\, S(i, r-1)$ for all $i \geq r \geq 1$.

Now, summing over i from r to $n - 1$, and noting that

$$\sum_{i=r}^{n-1} [S(i+1, r) - S(i, r)] = S(n, r) - S(r, r),$$

we get

$$S(n, r) - S(r, r) = \sum_{i=r}^{n-1} (i+1) S(i, r-1),$$

which now gives the desired result.

Example 2: Corollary 4 enables us to calculate $S(n, 2)$ for any integer $n \geq 2$. This is done as follows.

$$S(n, 2) = S(2, 2) + \sum_{i=2}^{n-1} (i+1) S(i,1) = \frac{1}{2} \sum_{i=1}^{n-1} i\,(i+1)^2$$

$$= \frac{1}{2} [\sum_{i=1}^{n-1} i^3 + 2 \sum_{i=1}^{n-1} i^2 + \sum_{i=1}^{n-1} i]$$

$$= \frac{1}{2} \left[\frac{n^2 (n-1)^2}{4} + \frac{n(n-1)(2n-1)}{3} + \frac{n(n-1)}{2} \right]$$

$$= \frac{(n-1) n (n+1)(3n+2)}{24},$$

the last expression follows after simplification.

Lemma 4: Let $n \geq 1$ be any integer. Then,

(i) $\sum_{r=0}^{n} l^{2r} S(2n, 2(n-r)) = \sum_{r=1}^{n} l^{2r-1} S(2n, 2(n-r)+1)$ for any $1 \leq l \leq 2n$,

(ii) $\sum_{r=0}^{n} l^{2r} S(2n+1, 2(n-r)+1) = \sum_{r=0}^{n} l^{2r+1} S(2n+1, 2(n-r))$ for any $1 \leq l \leq 2n+1$.

Proof: To prove the lemma, we make use of equation (1).

(i) From equation (1), for any $1 \leq l \leq 2n$, $G_{2n}(-l) = 0$. That is,

$$\sum_{r=0}^{2n} S(2n, 2n-r)(-l)^r = 0,$$

$$\sum_{r=0}^{2n} (-1)^r l^r S(2n, 2n-r) = 0. \tag{14}$$

Because of $\sum_{r=0}^{2n} = \sum_{r=0,2,4,\ldots 2n} + \sum_{r=1,3,5,\ldots,2n-1}$ the sum on the LHS of equation (14) can be written as

$$\sum_{r=0}^{2n} (-1)^r l^r S(2n, 2n-r) = \sum_{r=0}^{n} (-1)^{2r} l^{2r} S(2n, 2(n-r)) + \sum_{r=1}^{n} (-1)^{2r-1} l^{2r-1} S(2n, 2(n-r)+1),$$

which now gives the desired result.

(ii) In this case, we use the fact, $G_{2n+1}(-l) = 0$ for any $1 \le l \le 2n+1$. This gives

$$\sum_{r=0}^{2n+1} S(2n+1, 2n-r+1)(-l)^r = 0,$$

$$\sum_{r=0}^{2n+1} (-1)^r l^r S(2n+1, 2n-r+1) = 0.$$

Separating the even and the odd powers of l from the sum above, we get

$$\sum_{r=0}^{n} (-1)^{2r} l^{2r} S(2n+1, 2(n-r)+1) + \sum_{r=0}^{n} (-1)^{2r+1} l^{2r+1} S(2n+1, 2(n-r)) = 0,$$

which gives the required result.

Following result is found from Lemma 4 for the particular case when $l = 1$. This result is mentioned, in a different form, in Vyawahare (2006).

Corollary 5: For any integer $n \ge 1$,

(i) $\sum_{r=0}^{n} S(2n, 2(n-r)) = \sum_{r=1}^{n} S(2n, 2(n-r)+1)$,

(ii) $\sum_{r=0}^{n} S(2n+1, 2(n-r)+1) = \sum_{r=0}^{n} S(2n+1, 2(n-r))$.

Example 3: To verify Lemma 4 for $n = 2$ and $1 \le l \le 4$, we use the following values found in Vyawahare (2006).

$S(4, 0) = 1$, $S(4, 1) = 10$, $S(4, 2) = 35$, $S(4, 3) = 50$, $S(4, 4) = 24$.

In addition, we have found the values below.

$S(5, 0) = 1$, $S(5, 1) = 15$, $S(5, 2) = 85$, $S(5, 3) = 225$, $S(5, 4) = 274$, $S(5, 5) = 120$.

Now, since

$S(4, 4) + S(4, 2) + S(4, 0) = 60 = S(4, 3) + S(4, 1)$,

$S(5, 5) + S(5, 3) + S(5, 1) = 360 = S(5, 4) + S(5, 2) + S(5, 0)$,

Corollary 5 (which corresponds to $l = 1$ with $n = 2$) is verified.

Now, Lemma 4(i) with $l = 2$ gives

$$\sum_{r=0}^{2} 2^{2r} S(4, 4-2r) = \sum_{r=1}^{2} 2^{2r-1} S(4, 5-2r),$$

which is true, because

$S(4, 4) + 2^2 S(4, 2) + 2^4 S(4, 0) = 180 = 2S(4, 3) + 2^3 S(4, 1)$;

and Lemma 4(ii) with $l = 2$ becomes

$$\sum_{r=0}^{2} 2^{2r} S(5,5-2r) = \sum_{r=0}^{2} 2^{2r+1} S(5,4-2r),$$

which is also true, because

$S(5, 5) + 2^2 S(5, 3) + 2^4 S(5, 1) = 1260 = 2S(5, 4) + 2^3 S(5, 2) + 2^5 S(5, 0)$.

Similarly, corresponding to $l = 3$,

$S(4, 4) + 3^2 S(4, 2) + 3^4 S(4, 0) = 420 = 3S(4, 3) + 3^3 S(4, 1)$;

$S(5, 5) + 3^2 S(5, 3) + 3^4 S(5, 1) = 3360 = 3S(5, 4) + 3^3 S(5, 2) + 3^5 S(5, 0)$,

and finally, corresponding to $l = 4$,

$S(4, 4) + 4^2 S(4, 2) + 4^4 S(4, 0) = 840 = 4S(4, 3) + 4^3 S(4, 1)$;

$S(5, 5) + 4^2 S(5, 3) + 4^4 S(5, 1) = 7560 = 4S(5, 4) + 4^3 S(5, 2) + 4^5 S(5, 0)$.

Lemma 5: For any integers n and r with $n \geq r \geq 1$,

(i) $O(n + 1, r) = O(n, r) + (2n + 1)\, O(n, r - 1)$,

(ii) $E(n + 1, r) = E(n, r) + 2(n + 1)\, E(n, r - 1)$.

Proof: We prove part (i) only; the proof of part (ii) is similar. From equation (6), we have

$$(x+1)(x+3)\ldots(x+2n+1) = \sum_{r=0}^{n+1} O(n+1,r)x^{n+1-r} . \tag{15}$$

But,

LHS of equation (15) = $(x+2n+1)\sum_{r=0}^{n} O(n,r)x^{n-r}$

$$= \sum_{r=0}^{n} O(n,r)x^{n+1-r} + (2n+1)\sum_{r=0}^{n} O(n,r)x^{n-r}$$

$$= [O(n,0)\,x^{n+1} + \sum_{r=1}^{n} O(n,r)x^{n+1-r}\ ] + (2n+1)[\sum_{r=0}^{n-1} O(n,r)x^{n-r} + O(n,n)]$$

(replacing r by $r - 1$ in the second sum)

$$= O(n+1,0)\,x^{n+1} + \sum_{r=1}^{n}[O(n,r) + (2n+1)\,O(n,r-1)]x^{n+1-r} + O(n+1, n+1), \tag{16}$$

where we have made use of equation (4).

Comparing the coefficients of x^{n+1-r} from (15) and (16), we get the desired result.

Lemma 6: For any integers $n \geq 1$ and $p \geq 0$,

$$\sum_{r=0}^{n} S^{(p+1)}(n,r) = S^{(p+2)}(n, n).$$

Proof: From equation (8) with $x = 1$,

$$G_n^{(p+1)}(1) \equiv (p+2)(p+3)\ldots(p+n+1) = \sum_{r=0}^{n} S^{(p+1)}(n,r) . \tag{17}$$

Again, since

$$G_n^{(p+2)}(x) \equiv (x+p+2)(x+p+3)\ldots(x+p+n+1) = \sum_{r=0}^{n} S^{(p+2)}(n,r)x^{n-r} ,$$

it follows that

$$G_n^{(p+2)}(0) \equiv (p+2)(p+3)\ldots(p+n+1) = S^{(p+2)}(n, n). \tag{18}$$

Comparing equations (17) and (18), we get the result desired.

As a particular case of Lemma 6 with $p = 0$ (since $S^{(2)}(n, n) = (n + 1)!$), we have

$$\sum_{r=0}^{n} S(n,r) = (n+1)!,$$

which is just Corollary 1, proved earlier.

CONCLUSION

This study has presented some new properties of $S(n, r)$, $O(n, r)$, $E(n, r)$, and $S^{(p+1)}(n, r)$ provided in the results and discussion section. We have proved some recurrence formulae related to $S(n, r)$ mentioned, without any proof, in Vyawahare (2006) here. Two significant formulae related to Smarandache sum of products are demonstrated by numerical examples.

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