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ON VALUES OF Z(pq), q = mp ± 9

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ABSTRACT

The Smarandache function, denoted by $S(n)$, was introduced by Smarandache (1980), which is popular and well-studied. The next popular Smarandache type arithmetic function is the pseudo Smarandache function, introduced by Kashihara (1996), and is denoted by $Z(n)$. Though many researchers studied the function $Z(n)$, from the recent survey paper of Liu (2017), it appears that the explicit forms of $Z(n)$ are not abundant. Particularly few are the results on $Z(pq)$, where p and q are two distinct primes. This paper finds closed form expressions of $Z(pq)$, where p and $q>p$ are primes with q being of the forms $q=mp\pm9$, $m\geq1$ being an integer.

Keywords: Diophantine equation, pseudo Smarandache function, Smarandache function

INTRODUCTION

In the 1980's, the famous Romanian-American number theorist, Florentin Smarandache published some problems in the theory of number, some with answers, and the rest were open problems (Smarandache 1980). These problems are different from the traditional number theoretic problems. One prominent arithmetic function is $S(n)$, known as the Smarandache function. Soon after the publication of the problems, they drew the attention of the researchers all around the world.

Inspired by the works of Smarandache, Kashihara (1996) introduced the function, known as the pseudo Smarandache function, $Z(n)$, which is defined as follows:

$$Z(n) = min\left\{\lambda : \quad n | \frac{\lambda(\lambda+1)}{2}\right\}.$$

Kashihara (1996), Ashbacher (1998) and Majumdar (2010) derived the expressions of $Z(n)$ in some special cases of n, which are supplemented by Majumdar (2011), Majumdar (2012) and Majumdar and Islam (2019). The survey paper of Liu (2017) reviews the contributions of different researchers till 2016, which shows that the closed-form expressions of $Z(n)$ are only limited in number, and furthermore, no mention has been made of the expression of the form $Z(pq)$, where p and q are distinct odd primes. In this paper, we find $Z(pq)$, where $q>p$ for a limited case.

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MATERIALS AND METHODS

A method of finding $Z(pq)$ is given in Theorem 4.2.2 in Majumdar (2010), which would be employed in this paper, and hence, briefly reproduced below. Since

$$Z(pq) = min\left\{\lambda : \quad pq| \frac{\lambda(\lambda+1)}{2}\right\},$$

there are two possibilities as in Case 1 and Case 2.

Case 1: When p divides λ and q divides $\lambda+1$.

In this case, we get the following two equations:

$\lambda=pa$, $\lambda+1=qb$ for some integers $a\geq1$ and $b\geq1$.

This leads to the Diophantine equation below:

$$qb-pa=1. \tag{1}$$

Case 2: When p divides $\lambda+1$, q divides λ.

Then, we have

$\lambda+1=pa$, $\lambda=qb$ for some integers $a\geq1$ and $b\geq1$,

and the resulting Diophantine equation is

$$pa-qb=1. \tag{2}$$

Thus, in order to find $Z(pq)$, we need to solve the two Diophantine equations (1) and (2) for minimum values of a or b, as the case may be. This is done in results and discussion portion.

RESULTS AND DISCUSSION

This paper derives values of $Z(pq)$, for $q=mp+9$ ($m\geq2$), or $q=(m+1)p-9$ ($m\geq1$) (m being an integer) as follows.

1 Value of $Z(pq)$, $q=mp+9$

In this section, we find an expression of $Z(pq)$ when $q=mp+9$ for some integer $m\geq2$. This is given in the following Lemma 1.

When $q=mp+9$, $m\geq2$, the Diophantine equations (1) and (2) become

$(mp+9)b-pa=1$,

$pa-(mp+9)b=1$,

which may be rearranged as

$$9b-(a-mb)p=1, \tag{3}$$

$$(a-mb)p-9\text{b}=1. \tag{4}$$

If the minimum solution, say, (a_0, b_0) is found from (3), then by Case 1 in materials and methods section, the minimum λ_0 is

$$\lambda_0 = qb_0 - 1; \tag{5}$$

on the other hand, if (4) gives the minimum solution, say, (a_1, b_1), then by Case 2 in materials and methods, the minimum λ_1 is given by

$$\lambda_1 = qb_1. \tag{6}$$

The lemma below gives the closed-form expression of $Z(pq)$ when $q=mp+9$.

Lemma 1: Let $q=mp+9$, where $m\geq2$ is an integer. Then,

$$Z(pq)=\begin{cases}\frac{q(p-1)}{9}, & if\ \ 9\ \ divides\ \ (p-1)\\ \frac{q(4p+1)}{9}-1, & if\ \ 9\ \ divides\ \ (p-2)\\ \frac{q(2p+1)}{9}-1, & if\ \ 9\ \ divides\ \ (p-4)\\ \frac{q(2p-1)}{9}, & if\ \ 9\ \ divides\ \ (p-5)\\ \frac{q(4p-1)}{9}, & if\ \ 9\ \ divides\ \ (p-7)\\ \frac{q(p+1)}{9}-1, & if\ \ 9\ \ divides\ \ (p-8)\end{cases}.$$

Proof: We consider separately the following possible cases.
Case 1: When $p=9c+1$ for some integer $c\geq2$.
In this case, the Diophantine equations (3) and (4) become

$$1=9b-(a-mb)(9c+1)=9[b-(a-mb)c]-(a-mb),$$
$$1=(a-mb)(9c+1)-9b=(a-mb)-9[b-(a-mb)c].$$

The second of the above two equations gives the minimum solution with

$$a_1-mb_1=1,\ b_1-(a_1-mb_1)c=0.$$

Therefore,

$$b_1=c=\frac{p-1}{9},$$

and consequently,

$$\lambda_1=qb_1=qc=\frac{q(p-1)}{9}.$$

Case 2: When $p=9c+2$ for some integer $c\geq1$.
Then, equations (3) and (4) take the forms

$$1=9b-(a-mb)(9c+2)=9[b-(a-mb)c]-2(a-mb),$$
$$1=(a-mb)(9c+2)-9b=2(a-mb)-9[b-(a-mb)c].$$

The first of the above two Diophantine equations gives the minimum solution as follows:

$$a_0-mb_0=4,\ b_0-(a_0-mb_0)c=1,$$

so that $b_0=4c+1$. Now, since $p=9c+2$, we have

$$4p+1=9(4c+1).$$

Therefore,

$$\lambda_0=qb_0-1=q(4c+1)-1=\frac{q(4p+1)}{9}-1.$$

Case 3: When $p=9c+4$ for some integer $c\geq1$.
Here, since from (3) and (4), the Diophantine equations satisfied are

$$1=9b-(a-mb)(9c+4)=9[b-(a-mb)c]-4(a-mb),$$
$$1=(a-mb)(9c+4)-9b=4(a-mb)-9[b-(a-mb)c],$$

and the minimum solution, from the first equation, is

$$a_0-mb_0=2,\ b_0-(a_0-mb_0)c=1.$$

Therefore, $b_0=2c+1$. Now,

p=9c+4 $\Rightarrow$ 2p+1=9(2c+1),

and hence

$\lambda_0 = qb_0 - 1 = q(2c+1) - 1 = \frac{q(2p+1)}{9} - 1.$

Case 4: When p=9c+5 for some integer c≥0.

In this case, the Diophantine equations (3) and (4) become

$1=9b-(a-mb)(9c+5)=9[b-(a-mb)c]-5(a-mb),$

$1=(a-mb)(9c+5)-9b=5(a-mb)-9[b-(a-mb)c].$

Then, the second equation gives the minimum solution, with

$a_1 - mb_1 = 2$, $b_1 - (a_1 - mb_1)c = 1$.

Thus, $b_1 = 2c+1$. Since,

p=9c+5 $\Rightarrow$ 2p–1=9(2c+1),

we get

$\lambda_1 = qb_1 = q(2c+1) = \frac{q(2p-1)}{9}.$

Note that, in this case, when c=0, we get

$Z(5q)=q$ for $q=5k+9$; k=2, 4, …,

which is true by Lemma 4.2.17 in Majumdar (2010). Thus, the result is true for c=0 as well.

Case 5: When p=9c+7 for some integer c≥0.

Here, the Diophantine equations (3) and (4) reduce to

$1=9b-(a-mb)(9c+7)=9[b-(a-mb)c]-7(a-mb),$

$1=(a-mb)(9c+7)-9b=7(a-mb)-9[b-(a-mb)c],$

and we get the minimum solution from the second equation as follows:

$a_1 - mb_1 = 4$, $b_1 - (a_1 - mb_1)c = 3$.

Then, $b_1 = 4c+3$. To simplify, note that

p=9c+7 $\Rightarrow$ 4p–1=9(4c+3),

and hence,

$\lambda_1 = qb_1 = q(4c+3) = \frac{q(4p-1)}{9}.$

When c=0, we get

$Z(7q)=3q$, $q=7k+9$, k=2, 4, 6, …,

which is valid by Lemma 4.2.19 in Majumdar (2010), showing that the result holds for c=0 also.

Case 6: When p=9c+8 for some integer c≥1.

In this case, the Diophantine equations (3) and (4) take the form

$1=9b-(a-mb)(9c+8)=9[b-(a-mb)c]-8(a-mb),$

$1=(a-mb)(9c+8)-9b=8(a-mb)-9[b-(a-mb)c].$

Clearly, the first equation gives the minimum solution as follows:

$a_0 - mb_0 = 1$, $b_0 - (a_0 - mb_0)c = 1$.

Therefore,

$b_0 = c+1 = \frac{p+1}{9},$

and hence,

$\lambda_0 = qb_0 - 1 = q(c+1) - 1 = \frac{q(p+1)}{9} - 1.$

All these complete the proof of the lemma.

It may be mentioned here that, Lemma 1 may be recast in the following equivalent form.

Lemma 2: Let $q=mp+9$, where $m\geq 2$ is an integer. Then,

$$Z(pq)=\begin{cases} qa, & if \;\; p=9a+1 \\ q(4a+1)-1, & if \;\; p=9a+2 \\ q(2a+1)-1, & if \;\; p=9a+4 \\ q(2a+1), & if \;\; p=9a+5 \\ q(4a+3), & if \;\; p=9a+7 \\ q(a+1)-1, & if \;\; p=9a+8 \end{cases}.$$

Proof: Follows immediately from Lemma 1, and is left to the reader.

Lemma 2 needs some explanation. For example, if p is a prime of the form $p=9a+1$ for some integer a (≥ 1), and q is a prime of the form $q=mp+9$ for some integer m (≥ 2), then

$$Z(pq)=qa.$$

Thus, for instance, the first prime in the sequence is $p=19$ (corresponding to $a=2$). Note that, in Lemma 1 (and Lemma 2), m must be an even integer. Choosing $m=2$, the first prime of the form $q=mp+9$ is $q=47$ (corresponding to $m=2$). Thus, by Lemma 2,

$$Z(19\times 47)=Z(893)=94.$$

Again, corresponding to $p=11$, we get $Z(11\times 31)=Z(341)=154$.

2 Value of $Z(pq)$, $q=(m+1)p-9$

In this section, we derive $Z(pq)$, when $q=(m+1)p-9$ for some integer $m\geq 1$. The expression of $Z(pq)$ is given in Lemma 3, after making the necessary background preparation below.

With $q=(m+1)p-9$, the Diophantine equations (1) and (2) read as

$$[(m+1)p-9]b-pa=1,$$
$$pa-[(m+1)p-9]b=1,$$

that is,

$$1=[(m+1)b-a]p-9b, \tag{7}$$
$$1=9b-[(m+1)b-a]p. \tag{8}$$

Here also, the minimum solution (a_0, b_0) of (7) gives the minimum λ_0 as follows

$$\lambda_0=qb_0-1;$$

and the minimum solution (a_1, b_1) of (8) gives the minimum λ_1 as below

$$\lambda_1=qb_1.$$

The lemma below gives the expression of $Z(pq)$ in this case.

Lemma 3: Let $q=(m+1)p-9$, where $m\geq 3$ is an integer. Then,

$$Z(pq)=\begin{cases} \frac{q(p-1)}{9}-1, & if \;\; 9 \;\; divides \;\; (p-1) \\ \frac{q(4p+1)}{9}, & if \;\; 9 \;\; divides \;\; (p-2) \\ \frac{q(2p+1)}{9}, & if \;\; 9 \;\; divides \;\; (p-4) \\ \frac{q(2p-1)}{9}-1, & if \;\; 9 \;\; divides \;\; (p-5) \\ \frac{q(4p-1)}{9}-1, & if \;\; 9 \;\; divides \;\; (p-7) \\ \frac{q(p+1)}{9}, & if \;\; 9 \;\; divides \;\; (p-8) \end{cases}.$$

Proof: The following six possibilities may occur:
Case 1: When $p=9c+1$ for some integer $c\geq 2$.
In this case, the Diophantine equations (7) and (8) read as

$$1=[(m+1)b-a](9c+1)-9b=[(m+1)b-a]-9[b-\{(m+1)b-a\}c],$$
$$1=9b-[(m+1)b-a](9c+1)=9[b-\{(m+1)b-a\}c]-[(m+1)b-a].$$

The first of the above two equations gives the minimum solution, namely,

$$(m+1)b_0-a_0=1,\ b_0-\{(m+1)b_0-a_0\}c=0.$$

Therefore,

$$b_0=c=\frac{p-1}{9},$$

and hence

$$\lambda_0=qb_0-1=qc-1=\frac{q(p-1)}{9}-1.$$

Case 2: When $p=9c+2$ for some integer $c\geq 1$.
Here, (7) and (8) become

$$1=[(m+1)b-a](9c+2)-9b=2[(m+1)b-a]-9[b-\{(m+1)b-a\}c],$$
$$1=9b-[(m+1)b-a](9c+2)=9[b-\{(m+1)b-a\}c]-2[(m+1)b-a].$$

The second of the above two equations gives the minimum solution as follows:

$$(m+1)b_1-a_1=4,\ b_1-\{(m+1)b_1-a_1\}c=1.$$

Then, $b_1=4c+1$. Now, since

$$p=9c+2 \Rightarrow 4p+1=9(4c+1),$$

we get

$$\lambda_1=qb_1=q(4c+1)=\frac{q(4p+1)}{9}.$$

Case 3: When $p=9c+4$ for some integer $c\geq 1$.
In this case, from (7) and (8), we have

$$1=[(m+1)b-a](9c+4)-9b=4[(m+1)b-a]-9[b-\{(m+1)b-a\}c],$$
$$1=9b-[(m+1)b-a](9c+4)=9[b-\{(m+1)b-a\}c]-4[(m+1)b-a].$$

The minimum solution, from the second equation, is as follows:

$$(m+1)b_1-a_1=2,\ b_1-\{(m+1)b_1-a_1\}c=1.$$

Thus, $b_1=2c+1$. Since,

$$p=9c+4 \Rightarrow 2p+1=9(2c+1),$$

we get

$$\lambda_1=qb_1=q(2c+1)=\frac{q(2p+1)}{9}.$$

Case 4: When $p=9c+5$ for some integer $c\geq 0$.
From (7) and (8), we get

$$1=[(m+1)b-a](9c+5)-9b=5[(m+1)b-a]-9[b-\{(m+1)b-a\}c],$$
$$1=9b-[(m+1)b-a](9c+5)=9[b-\{(m+1)b-a\}c]-5[(m+1)b-a].$$

The first equation gives the minimum solution as below:

$$(m+1)b_0-a_0=2,\ b_0-\{(m+1)b_0-a_0\}c=1.$$

This gives $b_0=2c+1$. Noting that,

$$p=9c+5 \Rightarrow 2p-1=9(2c+1),$$

we get

$$\lambda_0 = qb_0 - 1 = q(2c+1) - 1 = \frac{q(2p-1)}{9} - 1.$$

It may be noted here that c=0 gives

$$Z(5q) = q-1;\ q = 5k-9,\ k = 4, 6, \ldots,$$

which is true by Lemma 4.2.17 in Majumdar (2010). Thus, the result is true for c=0 as well.

Case 5: When $p=9c+7$ for some integer $c \geq 0$.
Here, from (7) and (8), we have

$$1 = [(m+1)b - a](9c+7) - 9b = 7[(m+1)b - a] - 9[b - \{(m+1)b - a\}c],$$
$$1 = 9b - [(m+1)b - a](9c+7) = 9[b - \{(m+1)b - a\}c] - 7[(m+1)b - a].$$

The first equation gives the minimum solution as follows:

$$(m+1)b_0 - a_0 = 4,\ b_0 - \{(m+1)b_0 - a_0\}c = 3.$$

Therefore, $b_0 = 4c+3$, and since

$$p = 9c+7 \quad \Rightarrow \quad 4p-1 = 9(4c+3),$$

we finally get

$$\lambda_0 = qb_0 - 1 = q(4c+3) - 1 = \frac{q(4p-1)}{9} - 1.$$

In this case, for c=0, we get

$$Z(7q) = 3q-1;\ q = 7k-9,\ k = 4, 6, \ldots,$$

which is true by virtue of Lemma 4.2.19 in Majumdar (2010).

Case 6: When $p=9c+8$ for some integer $c \geq 1$.
From (7) and (8),

$$1 = [(m+1)b - a](9c+8) - 9b = 8[(m+1)b - a] - 9[b - \{(m+1)b - a\}c],$$
$$1 = 9b - [(m+1)b - a](9c+8) = 9[b - \{(m+1)b - a\}c] - 8[(m+1)b - a].$$

The second equation gives the minimum solution, with

$$(m+1)b_1 - a_1 = 1,\ b_1 - \{(m+1)b_1 - a_1\}c = 1.$$

That is,

$$b_1 = c+1 = \frac{p+1}{9},$$

and hence

$$\lambda_1 = qb_1 = q(c+1) = \frac{q(p+1)}{9}.$$

All these complete the proof of the lemma.

The equivalent alternative form of Lemma 3 is the following, whose proof is left as an exercise to the reader.

Lemma 4: Let $q=(m+1)p-9$, where $m \geq 3$ is an integer. Then,

$$Z(pq) = \begin{cases} qa - 1, & if\ \ p = 9a+1 \\ q(4a+1), & if\ \ p = 9a+2 \\ q(2a+1), & if\ \ p = 9a+4 \\ q(2a+1) - 1, & if\ \ p = 9a+5 \\ q(4a+3) - 1, & if\ \ p = 9a+7 \\ q(a+1), & if\ \ p = 9a+8 \end{cases}.$$

It may be mentioned here that, in Lemma 3 and Lemma 4, m must be an odd integer. It might be instructive to give some simple example from Lemma 3 (or, equivalently) Lemma 4. With p=19 (corresponding to *a*=1), the first prime satisfying the condition that *q*=2*p*–9 is *q*=29. Then, by Lemma 4, *Z*(19×29)=*Z*(551)=57.

Remarks

For the Smarandache function

$$S(n) = min\{m : n \;\; divides \;\; m!\},$$

it is known that

$$S(mn) = max\{S(m), S(n)\} \;\; if \;\; gcd(m,n) = 1.$$

Thus, if

$$n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \dots {p_r}^{\alpha_r}$$

is the representation of the integer *n* in terms of its *r* prime factors p_1, p_2, …, p_r, then the problem of finding *S*(*n*) reduces to the problem of finding $S({p_i}^{\alpha_i})$ where

$$S(n) = max\{S({p_i}^{\alpha_i}) : 1 \le i \le r\}.$$

However, the situation is different in case of the function *Z*(*n*), and it seems that *Z*(*n*) has to be found case-by-case. Possibly, this is the reason that in Smarandache (2000), the editor has advised us to study *Z*(*n*) in much more detail. In Majumdar (2010), the closed-form expressions of *Z*(*pq*) with primes *p* and *q* with *q*=*mp*±*l*, 1≤*l*≤8, are given. This paper extends those results to include the cases when *q* is of the forms *q*=*mp*±9.

It may be mentioned here that, Ibstedt (1997) (see also Majumdar 2010) proposes an alternative procedure to find *Z*(*pq*). However, from the point of view of application of the procedure, the method outlined here is simpler. In simple cases, the solutions of the Diophantine equations (1) and (2) can be found by inspection.

As an illustration of the application of Lemma 1, we give below the closed-form expressions of *Z*(*pq*) for some particular values of *p*:

(1) *Z*(19*q*)=2*q*; *q*=19*m*+9, *m*=2, 4, 6, …,
(2) *Z*(11*q*)=5*q*–1; *q*=11*m*+9, *m*=2, 4, 6, …,
(3) *Z*(13*q*)=3*q*–1; *q*=13*m*+9, *m*=2, 4, 6, …,
(4) *Z*(23*q*)=5*q*; *q*=23*m*+9, *m*=2, 4, 6, …,
(5) *Z*(97*q*)=43*q*; *q*=97*m*+9, *m*=2, 4, 6, …,
(6) *Z*(17*q*)=2*q*–1; *q*=17*m*+9, *m*=2, 4, 6, … .

Note that, for a given *p*, the distribution of the corresponding *q* is quite sparse. For example, in Case (1), the first such *q* is *q*=47, as has been mentioned before. However, the next such *q* is *q*=199 (corresponding to *m*=10). The next such *p* is *p*=37 with the corresponding *q*=83. In Case (2) above, the first *p* is *p*=11 with the corresponding *q*=31. In Case (3), the first *p* is *p*=13, and the corresponding *q* is *q*=61, in Case (4), the first *p*=23 with *q*=101, in Case (5), we see that, the first *p* is *p*=97 (corresponding to *a*=10) with the corresponding *q*= 397. And in Case (6), the first *p* is *p*=17 with the corresponding *q*=43.

From Lemma 3, we may derive the following six expressions:

(1) *Z*(19*q*)=2*q*–1; *q*=19*k*–9, *k*=2, 4, 6, …,
(2) *Z*(11*q*)=5*q*; *q*=11*k*–9, *k*=2, 4, 6, …,
(3) *Z*(29*q*)=13*q*; *q*=29*k*–9, *k*=2, 4, 6, …,

(4) $Z(13q)=3q$; $q=13k-9$, $k=2, 4, 6, \ldots$,
(5) $Z(23q)=5q-1$; $q=23k-9$, $k=2, 4, 6, \ldots$,
(6) $Z(17q)=2q$; $q=17k-9$, $k=2, 4, 6, \ldots$.

In this case, for example, in Case (1), corresponding to the first prime $p=19$, the first prime q of the form $q=(m+1)p-9$ is $q=29$, so that $Z(19\times 29)=Z(551)=57=2q-1$. Again, in Case (2), the first prime is $p=11$ with the corresponding $q=13$, and $Z(143)=65=5q$. In Case (3), for $p=29$, the corresponding q is $q=107$ with $Z(3103)=1391=13q$. In Case (4), the first p is $p=13$, with $q=17$ and $Z(221) = 51 =3q$. In Case (5), corresponding to the first prime $p = 23$ is $q = 37$, with $Z(851)=184=5q-1$. And finally, in Case (6), for $p=17$, the corresponding q is found to be $q=59$, with $Z(1003)=118=2q$.

It is conjectured that, in general (if n is not of the form 2^m),

$$Z(n)\leq n-1.$$

Both Lemma 1 and Lemma 3 support the conjecture. Moreover, for example, from Case (1) of Lemma 1 and Lemma 2, we see that the values of $Z(19q)$ differ, depending on the form of the prime q itself. Thus, as has already been pointed out, to find the values of $Z(n)$ in general, we have to consider the form of n case-by-case.

Though the study of the function $Z(n)$ is still in infancy, it reveals interesting characteristics. For example, the Diophantine equation

$$Z(n)=Z(n+1)$$

has no solution, but the equation

$$Z(2n)=Z(3n)=Z(6n)$$

possesses a solution (namely, $n=11$)! We hope that more research on $Z(n)$ would reveal more interesting features of this new arithmetic function. For example, given a prime p (≥ 2), let the prime q be of the form $q=mp+k$ for some integers m (≥ 2) and k (≥ 2). Then, is it true that the equation

$$Z(pq) = \frac{q}{k} Z(p)$$

always possesses a solution?

CONCLUSION

This paper has studied the pseudo Smarandache function $Z(n)$, which is a popular Smarandache type arithmetic function. It has derived the closed form expressions of $Z(pq)$, where p and $q>p$ are primes with q being of the forms $q=mp\pm 9$, $m\geq 1$ being an integer. Theories are illustrated with some numerical data. Further research on pseudo Smarandache function considering the case of $q = mp + k$ ($m\geq 1$ and $k\geq 1$ are integers) is identified for the future researchers.

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